Chapter 7:
Linear Transformations

Ideas in this section…

1) Definition and examples of linear transformations
2) Some properties of Linear Transformations
3) Kernel and Image of a linear transformation
4) Examples of Kernel and Image of a linear transformation
5) Rank – Nullity Theorem
6) Isomorphisms

Definition and Examples of Linear Transformations

Def: If V and W are 2 vectors spaces, a function $T: V \to W$ is called a linear transformation if

1) $T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2)$ for all $\vec{v}_1, \vec{v}_2 \in V$

2) $T(r\vec{v}) = r\,T(\vec{v})$ for all $\vec{v} \in V$ and $r \in \mathbb{R}$

Definition and Examples of Linear Transformations

Ex 1: Show that the following are linear transformations…

a) $T: M_{22} \to P_2$ given by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = c + (b + d)x^2$

b) If $a \in \mathbb{R}$, $E_a: P_n \to \mathbb{R}$ given by $E_a\big(p(x)\big) = p(a)$

c) $D: P_n \to P_{n-1}$ given by $D\big(p(x)\big) = p'(x)$

Some Properties of Linear Transformations

Theorem 7.1.1

Let $T : V \to W$ be a linear transformation.

1. $T(\mathbf{0}) = \mathbf{0}$.
2. $T(-\mathbf{v}) = -T(\mathbf{v})$ *for all* $\mathbf{v}$ *in* V.
3. $T(r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k) = r_1T(\mathbf{v}_1) + r_2T(\mathbf{v}_2) + \cdots + r_kT(\mathbf{v}_k)$ *for all* $\mathbf{v}_i$ *in* V *and all* r_i *in* $\mathbb{R}$.

Proof:

Kernel and Image of a Linear Transformation

Def: Let V and W be vector spaces and let $T: V \rightarrow W$ be a linear transformation. Then the kernel of T and the image of T are defined by…

$$\ker T = \{ \vec{v} \in V \mid T(\vec{v}) = \vec{0} \}$$

$$im\, T = \{ T(\vec{v}) \mid \vec{v} \in \mathbb{R} \}$$

Kernel and Image of a Linear Transformation

Theorem 7.2.1

Let $T : V \to W$ be a linear transformation.

1. $\ker T$ *is a subspace of* V.
2. $\operatorname{im} T$ *is a subspace of* W.

Proof:

Examples of Kernel & Image of a Linear Transformation

Ex 2: Find the kernel and image of each of these linear transformations…

a) $T: M_{22} \to P_2$ given by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = c + (b+d)x^2$

b) If $a \in \mathbb{R}$, $E_a: P_n \to \mathbb{R}$ given by $E_a\big(p(x)\big) = p(a)$

c) $D: P_n \to P_{n-1}$ given by $D\big(p(x)\big) = p'(x)$

Examples of Kernel & Image of a Linear Transformation

Ex 2: Find the kernel and image of each of these linear transformations…

b) If $a \in \mathbb{R}$, $E_a : P_n \to \mathbb{R}$ given by $E_a\big(p(x)\big) = p(a)$

Examples of Kernel & Image of a Linear Transformation

Ex 2: Find the kernel and image of each of these linear transformations…

c) $D: P_n \to P_{n-1}$ given by $D\big(p(x)\big) = p'(x)$

Rank – Nullity Theorem

Def: Let V and W be vector spaces and let $T: V \to W$ be a linear transformation. Then the rank of T and the nullity of T are defined by…

$$\text{nullity}(T) = \dim(\ker T)$$

$$rank(T) = \dim(im\, T)$$

Recall: If A is an $m \times n$ matrix, we can define a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ by $T(\vec{x}) = A\vec{x}$. Then…(kernel, image, dimensions, rank-nullity)

Rank – Nullity Theorem

Thm (Rank-Nullity Theorem): Let V and W be vector spaces and let $T: V \to W$ be a linear transformation. Assume $\ker T$, $im\ T$, and V are finite dimensional. Then…

$$\dim(im\ T) + \dim(\ker T) = \dim V$$

or

$$rank(t) + nullity(t) = \dim V$$

Proof:

Isomorphisms

Def: A function $T: V \to W$ is one-to-one if…

$$\forall \ \vec{v}_1, \vec{v}_2 \in V \text{ , if } T(\vec{v}_1) = T(\vec{v}_2) \text{ , then } \vec{v}_1 = \vec{v}_2$$

Meaning…

Isomorphisms

Def: A function $T: V \to W$ is onto if…

$$\forall \ \vec{w} \in W \ , \ \exists \ \vec{v} \in V \text{ such that } T(\vec{v}) = \vec{w}$$

Meaning…

Isomorphisms

Def: A function $T: V \to W$ is an isomorphism if…

1) T is a linear transformation,
2) T is one-to-one, and
3) T is onto

Meaning…

Isomorphisms

Thm: A linear transformation $T: V \to W$ is one-to-one if and only if

$$\ker T = \{\vec{0}\}$$

Proof:

Isomorphisms

Ex 3: Show that the following linear transformations are isomorphisms…

a) $T: \mathbb{R}^2 \to P_1$ defined by $T(a, b) = a + bx$

b) $T: M_{22} \to P_3$ defined by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = c + ax + (a + b)x^2 + 2dx^3$

What you need to know from the book

Book reading

Sec. 7.1: pg. 375 – top half of 378

Sec. 7.2: pg. 382 – 387

Sec. 7.3: pg. 392

Problems you need to know how to do from the book

Sec. 7.1: #'s 1-3, 6, 8-9, 15, 17

Sec. 7.2: #'s 1-4, 6-11, 14-15

Sec. 7.3: #'s 1